Three-phase systems

1. Introduction

Three-phase systems are commonly used in generation, transmission and distribution of electric power. Power in a three-phase system is constant rather than pulsating and three-phase motors start and run much better than single-phase motors. A three-phase system is a generator-load pair in which the generator produces three sinusoidal voltages of equal amplitude and frequency but differing in phase by 120° from each other.
The phase voltages $v_a(t)$, $v_b(t)$ and $v_c(t)$ are as follows

$$\begin{aligned} v_a &= V_m \cos \omega t \\ v_b &= V_m \cos\left(\omega t - 120^\circ\right) \\ v_c &= V_m \cos\left(\omega t - 240^\circ\right), \end{aligned} \tag{1}$$

whereas the corresponding phasors are

$$\begin{aligned} V_a &= V_m \\ V_b &= V_m e^{-j120^\circ} \\ V_c &= V_m e^{-j240^\circ} \ . \end{aligned} \tag{2}$$

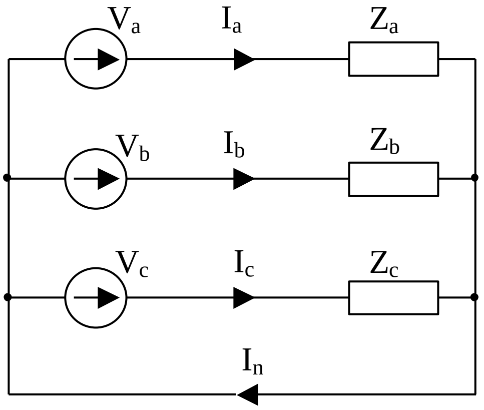

Fig.1

A three-phase system is shown in Fig 1. In a special case all impedances are identical

$$Z_a = Z_b = Z_c = Z \ . \tag{3}$$

Such a load is called a balanced load and is described by equations

$$I_a = \frac{V_a}{Z} \qquad I_b = \frac{V_b}{Z} \qquad I_c = \frac{V_c}{Z} \ .$$

Using KCL, we have

$$I_n = I_a + I_b + I_c = \frac{1}{Z}\left(V_a + V_b + V_c\right) , \qquad (4)$$

where

$$V_a + V_b + V_c = V_m\left(1 + e^{-j120^\circ} + e^{-j240^\circ}\right) =$$
$$= V_m\left(1 + \cos 120^\circ - j\sin 120^\circ + \cos 240^\circ - j\sin 240^\circ\right) = V_m\left(1 - \frac{1}{2} - j\frac{\sqrt{3}}{2} - \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = 0 .$$

Setting the above result into (4), we obtain

$$I_n = 0 \ . \qquad (5)$$

Since the current flowing though the fourth wire is zero, the wire can be removed (see Fig.2)

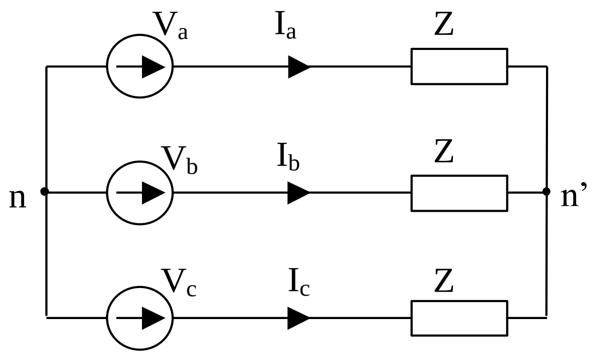

Fig. 2

The system of connecting the voltage sources and the load branches, as depicted in Fig. 2, is called the Y system or the star system. Point n is called the neutral point of the generator and point n' is called the neutral point of the load.

Each branch of the generator or load is called a phase. The wires connecting the supply to the load are called the lines. In the Y-system shown in Fig. 2 each line current is equal to the corresponding phase current, whereas the line-to-line voltages (or simply line voltages) are not equal to the phase voltages.

2 Y-connected systems

Now we consider the Y-connected generator sources (see Fig. 3).

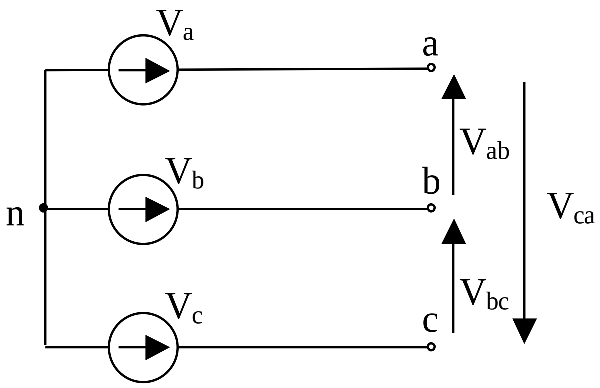

Fig.3

The phasors of the phase voltages can be generally written as follows

$$\begin{aligned} V_a &= V = V_m e^{j\alpha} \\ V_b &= V e^{-j120^o} \\ V_c &= V e^{-j240^o} \end{aligned} \quad . \tag{6}$$

We determine the line voltages V_{ab}, V_{bc}, V_{ca} (see Fig.3). Using KVL, we obtain

$$V_{ab} = V_a - V_b = V_a\left(1 + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = V_a\left(\frac{3}{2} + j\frac{\sqrt{3}}{2}\right) =$$

$$= V_a\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, e^{j\tan^{-1}\frac{\sqrt{3}}{3}} = V_a\sqrt{3}e^{j30^\circ} \; .$$

Thus,

$$V_{ab} = V_a\sqrt{3}e^{j30^o} \; . \tag{7}$$

holds and similarly we obtain

$$V_{bc} = V_b\sqrt{3}e^{j30^o} \tag{8}$$

$$V_{ca} = V_c\sqrt{3}e^{j30^o} \; . \tag{9}$$

The phasor diagram showing the phase and line voltages is shown in Fig.4.

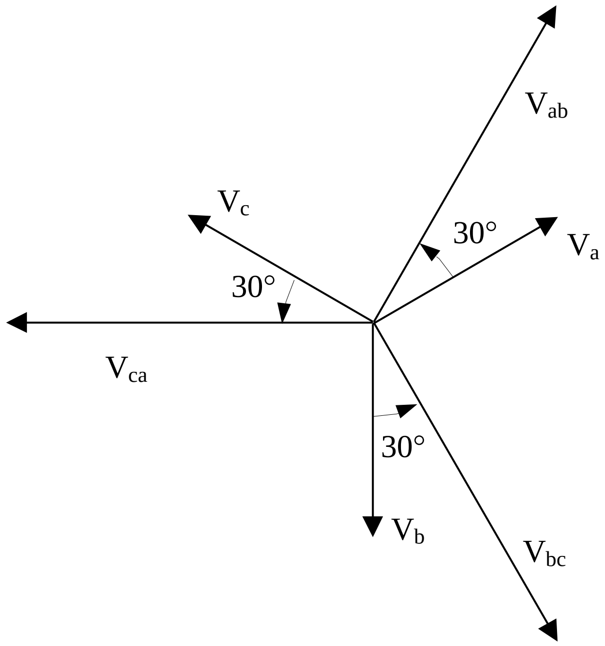

Fig.4

Thus, the line voltages V_{ab}, V_{bc}, V_{ca} form a symmetrical set of phasors leading by 30° the set representing the phase voltages and they are $\sqrt{3}$ times greater.

$$|V_{ab}| = |V_{bc}| = |V_{ca}| = \sqrt{3}|V_a| \;. \tag{10}$$

The same conclusion is valid in the Y connected load (see Fig.5).

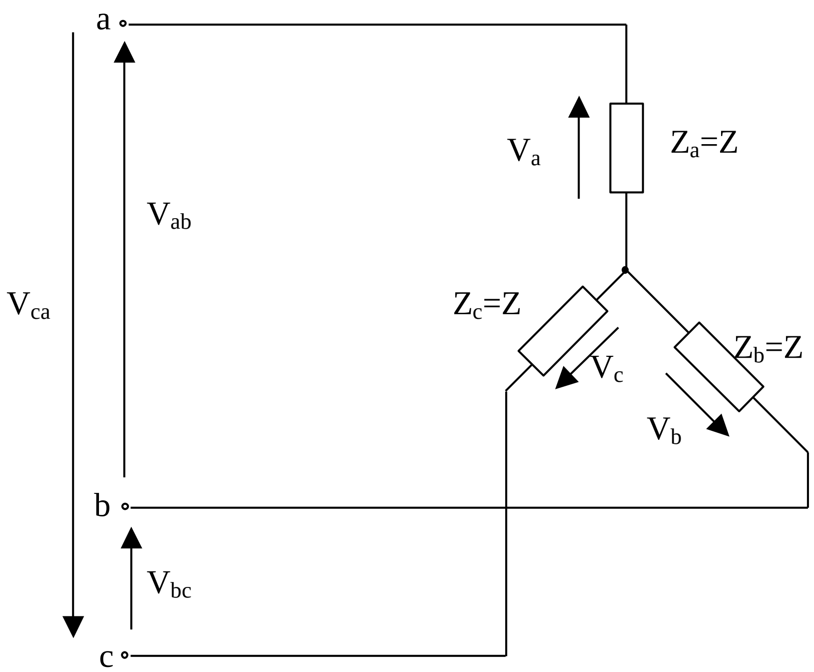

Fig.5

3. Three-phase systems calculations

When the three phases of the load are not identical, an unbalanced system is produced. An unbalanced Y-connected system is shown in Fig.1. The system of Fig.1 contains perfectly conducting wires connecting the source to the load. Now we consider a more realistic case where the wires are represented by impedances Z_p and the neutral wire connecting n and n' is represented by impedance Z_n (see Fig.6).

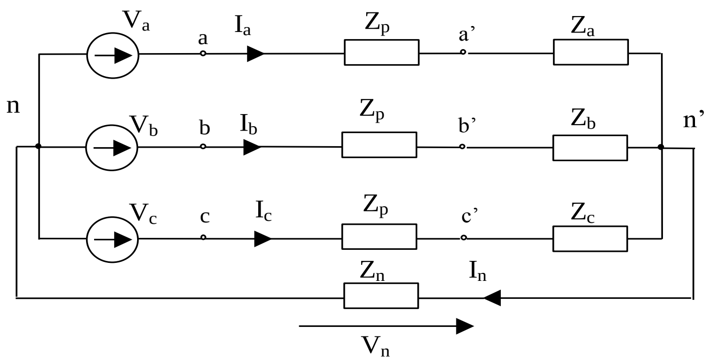

Fig.6

Using the node n as the datum, we express the currents I_a, I_b, I_c and I_n in terms of the node voltage V_n

$$I_a = \frac{V_a - V_n}{Z_a + Z_p} \tag{11}$$

$$I_b = \frac{V_b - V_n}{Z_b + Z_p} \tag{12}$$

$$I_c = \frac{V_c - V_n}{Z_c + Z_p} \tag{13}$$

$$I_n = \frac{V_n}{Z_n} \ . \tag{14}$$

Hence, we obtain the node equation

$$\frac{V_n}{Z_n} - \frac{V_a - V_n}{Z_a + Z_p} - \frac{V_b - V_n}{Z_b + Z_p} - \frac{V_c - V_n}{Z_c + Z_p} = 0$$

Solving this equation for V_n, we have

$$V_n = \frac{\dfrac{V_a}{Z_a + Z_p} + \dfrac{V_b}{Z_b + Z_p} + \dfrac{V_c}{Z_c + Z_p}}{\dfrac{1}{Z_n} + \dfrac{1}{Z_a + Z_p} + \dfrac{1}{Z_b + Z_p} + \dfrac{1}{Z_c + Z_p}} \ . \tag{15}$$

The above relationships enable us to formulate a method for the analysis of three-phase systems. The method consists of three steps as follows:

(i) Determine V_n using (15)
(ii) Calculate the currents I_a, I_b, I_c and I_n applying (11) - (14).
(iii) Find the phase and line voltages using Kirchhoff's and Ohm's laws.

When the neutral wire is removed, the system contains three connecting wires and is called a three-wire system. In such a case we set $|Z_n| \to \infty$ into (15)

$$V_n = \frac{\dfrac{V_a}{Z_a + Z_p} + \dfrac{V_b}{Z_b + Z_p} + \dfrac{V_c}{Z_c + Z_p}}{\dfrac{1}{Z_a + Z_p} + \dfrac{1}{Z_b + Z_p} + \dfrac{1}{Z_c + Z_p}} \ . \qquad (16)$$

The balanced system can be considered as a special case of the unbalanced system, where $Z_a = Z_b = Z_c = Z$. Using (16), we obtain

$$V_n = \frac{\dfrac{1}{Z + Z_p}\left(V_a + V_b + V_c\right)}{\dfrac{3}{Z + Z_p}} = 0 \ . \qquad (17)$$

Consequently, the relationships (11)-(13) reduce to

$$I_a = \frac{V_a}{Z + Z_p} \qquad (18)$$

$$I_b = \frac{V_b}{Z + Z_p} \qquad (19)$$

$$I_c = \frac{V_c}{Z + Z_p} \ . \qquad (20)$$

Since $V_b = V_a e^{-j120^\circ}$ and $V_c = V_a e^{-j240^\circ}$, we have $I_b = I_a e^{-j120^\circ}$ and $I_c = I_a e^{-j240^\circ}$.
Hence, we need to calculate I_a only using (18), which can be made applying the one-phase circuit described by equation (18) shown in Fig.7.

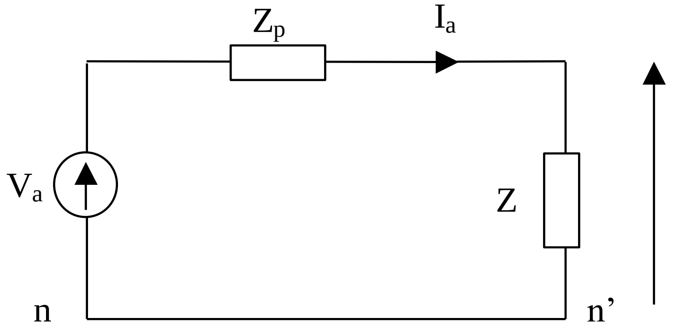

Fig.7

This means that the analysis of a balanced three-phase system can be reduced to the analysis of one-phase system depicted in Fig.7.

Example

Let us consider three-phase system shown in Fig.8. The system is supplied with a balanced three-phase generator, whereas the load is unbalanced.

The effective value of the generator phase voltage is 220V, the impedance of any connecting wire is $Z_p = (2 + j2)\Omega$ and the phase impedances of the load are $Z_a = (2 + j4)\Omega$, $Z_b = (4 - j2)\Omega$, $Z_c = (2 + j4)\Omega$. We wish to determine the line currents.

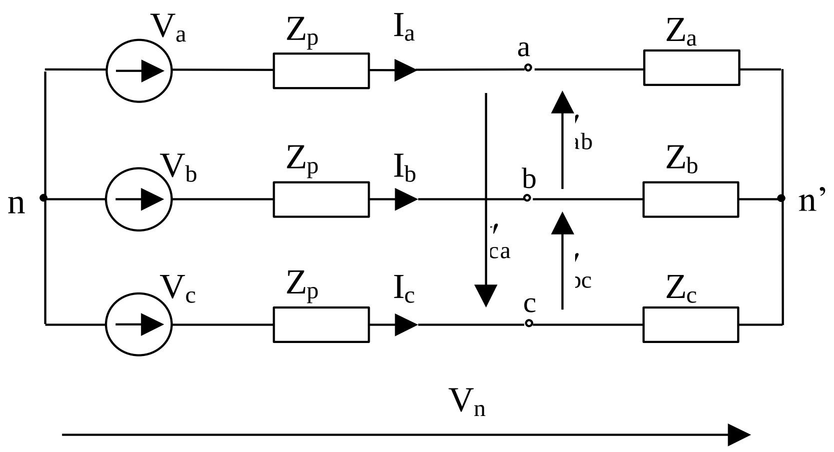

Fig.8

Since the circuit of Fig.8 is a three-wire system, we apply equation (16) to compute V_n. The phase generator voltages are

$$V_a = 220\sqrt{2}V$$

$$V_b = V_a e^{-j120^\circ} = 220\sqrt{2}\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = (-155.56 - j269.44)V$$

$$V_c = V_a e^{-j240^\circ} = 220\sqrt{2}\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = (-155.56 + j269.44)V\,.$$

Using (16), we find

$$V_n = \frac{\frac{220\sqrt{2}}{4 + j6} + \frac{(-155.56 - j269.44)}{6} + \frac{(-155.56 + j269.44)}{4 + j6}}{\frac{1}{4 + j6} + \frac{1}{6} + \frac{1}{4 + j6}} = (97.5 - j61.2)V\,.$$

Next, we compute the line currents using (11)-(13)

$$I_a = \frac{V_a - V_n}{Z_a + Z_p} = \frac{220\sqrt{2} - 97.5 + j61.2}{2 + j4 + 2 + j2} = (23.49 - j19.94)A$$

$$I_b = \frac{V_b - V_n}{Z_b + Z_p} = \frac{-155.56 - j269.44 - 97.5 + j61.2}{4 - j2 + 2 + j2} = (-42.18 - j34.70)A$$

$$I_c = \frac{V_c - V_n}{Z_c + Z_p} = \frac{-155.56 + j269.44 - 97.5 + j61.2}{2 + j4 + 2 + j2} = (18.68 + j54.63)A\ .$$

4 Power in three-phase circuits

In the balanced systems, the average power consumed by each load branch is the same and given by

$$\tilde{P}_{av} = V_{eff} I_{eff} \cos\phi \tag{21}$$

where V_{eff} is the effective value of the phase voltage, I_{eff} is the effective value of the phase current and ϕ is the angle of the impedance. The total average power consumed by the load is the sum of those consumed by each branch, hence, we have

$$P_{av} = 3\tilde{P}_{av} = 3V_{eff} I_{eff} \cos\phi \tag{22}$$

In the balanced Y systems, the phase current has the same amplitude as the line current $I_{eff} = (I_{eff})_L$, whereas the line voltage has the effective value $(V_{eff})_L$ which is $\sqrt{3}$ times greater than the effective value of the phase voltage, $(V_{eff})_L = \sqrt{3}V_{eff}$. Hence, using (22), we obtain

$$P_{av} = 3\frac{(V_{eff})_L}{\sqrt{3}}(I_{eff})_L \cos\phi = \sqrt{3}(V_{eff})_L (I_{eff})_L \cos\phi \tag{23}$$

Similarly, we derive

$$P_X = \sqrt{3}(V_{eff})_L (I_{eff})_L \sin\phi\ . \tag{24}$$

In the unbalanced systems, we add the powers of each phase

$$P_{av} = (V_{eff})_a (I_{eff})_a \cos\phi_a + (V_{eff})_b (I_{eff})_b \cos\phi_b + (V_{eff})_c (I_{eff})_c \cos\phi_c \tag{25}$$

$$P_X = (V_{eff})_a (I_{eff})_a \sin\phi_a + (V_{eff})_b (I_{eff})_b \sin\phi_b + (V_{eff})_c (I_{eff})_c \sin\phi_c\ . \tag{26}$$

In order to measure the average power in a three-phase Y-connected load, we use three wattmeters connected as shown in Fig.9.
The reading of the wattmeter W_1 is

$$P_{W_1} = \frac{1}{2}\mathrm{Re}\left(V_a I_a^*\right) = \frac{1}{2}(V_m)_a (I_m)_a \cos\phi_a = (V_{eff})_a (I_{eff})_a \cos\phi_a = P_a\ .$$

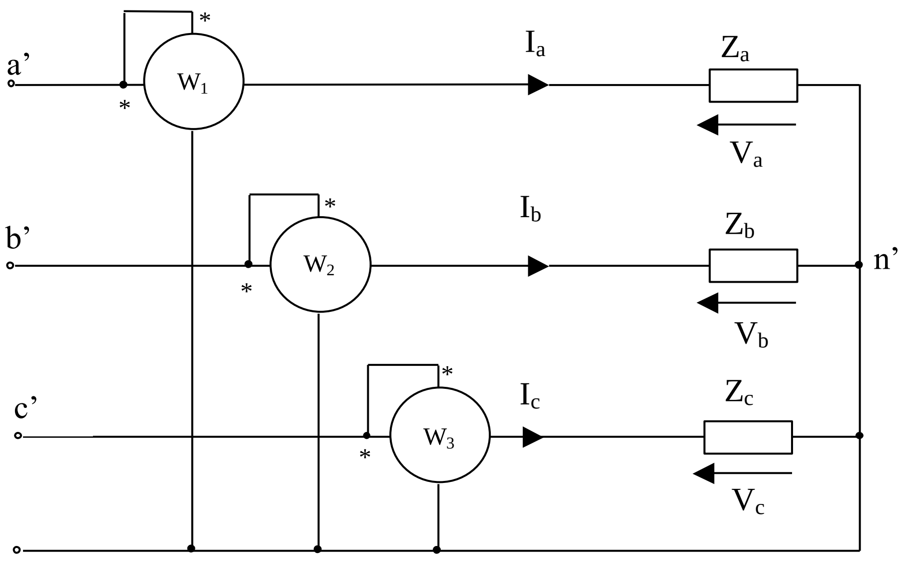

Fig. 9

Similarly, W_2 and W_3 measure the average power of the load branch b and c, respectively. Thus, the sum of the three readings will give the total average power. This method of the average power measurement is valid for both balanced and unbalanced Y-connected loads. Note that in the case of a balanced Y-connected load all three readings are identical and therefore we use only one wattmeter.

For measuring average power in a three-phase three-wire system, we can use a method exploiting two wattmeters. In this method two wattmeters are connected by choosing any one line as the common reference for the voltage coils of the wattmeters. The current coils are connected in series with the other two lines (see Fig.10) and the asterisk terminals of each wattmeter are short-circuited (see Fig.10).

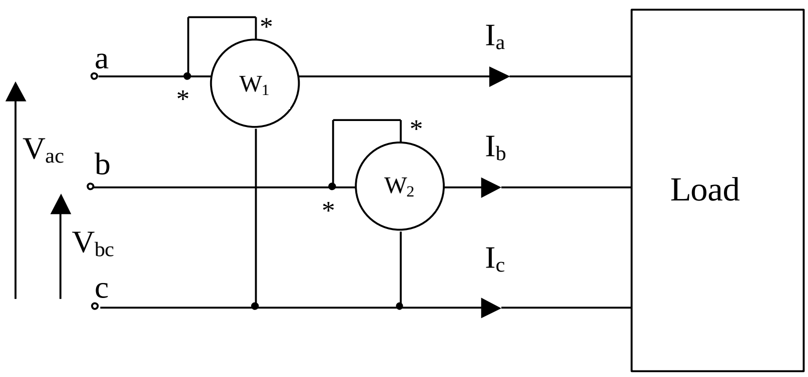

Fig.10

The indications of the wattmeters are

$$P_{W_1} = \frac{1}{2}\mathrm{Re}\left(V_{ac} I_a^*\right) \, , \tag{27}$$

$$P_{W_2} = \frac{1}{2}\mathrm{Re}\left(V_{bc} I_b^*\right) \, . \tag{28}$$

The load is shown in Fig.11.

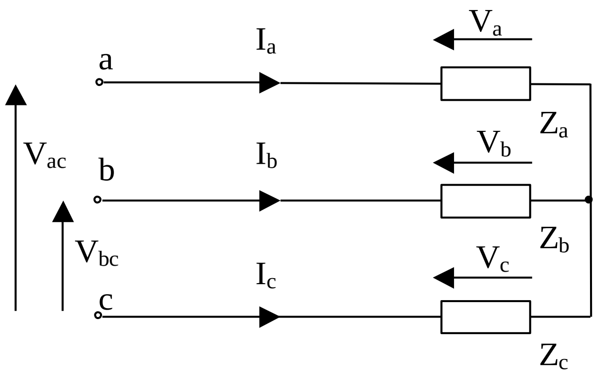

Fig.11

Since $V_{ac} = V_a - V_c$ and $V_{bc} = V_b - V_c$, we obtain

$$P_{W_1} = \frac{1}{2}\mathrm{Re}\left((V_a - V_c)I_a^*\right) = \frac{1}{2}\left(V_a I_a^* - V_c I_a^*\right),$$
$$P_{W_2} = \frac{1}{2}\mathrm{Re}\left((V_b - V_c)I_b^*\right) = \frac{1}{2}\left(V_b I_b^* - V_c I_b^*\right).$$

The sum of P_{W_1} and P_{W_2} gives

$$P_{W_1} + P_{W_2} = \frac{1}{2}\mathrm{Re}\left[V_a I_a^* + V_b I_b^* - V_c\left(I_a^* + I_b^*\right)\right]. \quad (29)$$

Currents I_a, I_b, I_c satisfy KCL

$$I_a + I_b + I_c = 0$$

Hence, it holds

$$I_a^* + I_b^* + I_c^* = 0 ,$$

or

$$I_a^* + I_b^* = - I_c^* . \quad (30)$$

Substituting (30) into (29) we have

$$P_{W_1} + P_{W_2} = \frac{1}{2}\mathrm{Re}\left[V_a I_a^* + V_b I_b^* + V_c I_c^*\right] = P_{av} . \quad (31)$$

Equation (31) says that the sum of the two wattmeters readings in a Y-connected system equals the total average power consumed by the load.

Let us consider a balanced Y-connected load and calculate the instantaneous power delivered by the generator to the load

$$p(t) = v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t) , \quad (32)$$

where

$$
\begin{aligned}
v_a(t) &= V_m \cos\omega t \\
v_b(t) &= V_m \cos\left(\omega t - 120^o\right) \\
v_c(t) &= V_m \cos\left(\omega t - 240^o\right)
\end{aligned}
\qquad (33)
$$

and

$$
\begin{aligned}
i_a(t) &= V_m \cos(\omega t - \phi) \\
i_b(t) &= V_m \cos\left(\omega t - 120^\circ - \phi\right) \\
i_c(t) &= V_m \cos\left(\omega t - 240^\circ - \phi\right).
\end{aligned}
\qquad (34)
$$

where $v_a(t), v_b(t), v_c(t)$ are the voltages of the load branches, $i_a(t), i_b(t), i_c(t)$ are the currents of the load branches and ϕ is the angle of the load impedance. We substitute (33)-(34) in (32)

$$
\begin{aligned}
p(t) = V_m I_m [&\cos\omega t \cos(\omega t - \phi) + \cos\left(\omega t - 120^o\right)\cos\left(\omega t - 120^o - \phi\right) + \\
&+ \cos\left(\omega t - 240^o\right)\cos\left(\omega t - 240^o - \phi\right)]
\end{aligned}
$$

and use the trigonometric identity

$$
\cos x \cdot \cos y = \frac{1}{2}\left[\cos(x - y) + \cos(x + y)\right] ,
$$

finding

$$
p(t) = \frac{1}{2} V_m I_m \left[3\cos\phi + \cos(2\omega t - \phi) + \cos\left(2\omega t - 240^o - \phi\right) + \cos\left(2\omega t - 480^o - \phi\right)\right] .
$$

Since

$$
\cos(2\omega t - \phi) + \cos\left(2\omega t - 240^o - \phi\right) + \cos\left(2\omega t - 480^o - \phi\right) = 0
$$

we obtain

$$
p(t) = \frac{3}{2} V_m I_m \cos\phi = 3 V_{eff} I_{eff} \cos\phi = P_{av} \qquad (35)
$$

Thus, the total instantaneous power p(t) delivered by a three-phase generator to the balanced load is constant and equals the average power consumed by the load.